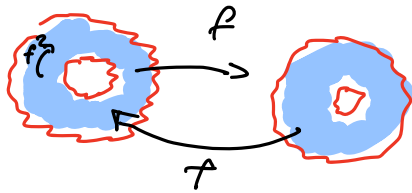


Classification of Fatou Components

Def. A Herman ring is a Fatou component isomorphic to an annulus $\{1 < |z| < e^{2\pi n}\}$ on which f^n is conjugate to rotation by an irrational angle for some n .



cycle of Herman rings

Show Fatou set is a union of
Herman rings

Prop. If U is a Herman ring, then
 $\partial U \subset \overline{P}(f)$.

Fatou component:



Preperiodic: finite
forward orbit

Wandering: infinite
forward orbit

Thm. (Sullivan) Rational maps have no
wandering domains.

Thm. A Every Fatou component is a
branched cover of

- 1) an immediate attracting basin
- 2) an immediate parabolic basin
- 3) a Siegel disk
- 4) a Herman ring

else: $P_{\text{preimage}} \rightarrow f(u) = u$, u is hyperbolic

Thm. B. X hyp. R.S., $f: X \rightarrow X$ holo.

Precisely one of the following hold:

- 1) (Attracting) f has an attracting fixed point. All points of X converge locally unif. to the fixed point.
- 2) (Escape) There exists an orbit w/ no accumulation point. Then no orbit has an accumulation point.

Moreover, if $X \subset Y$ for Y compact and f extends to ∂X w/ only finitely many fixed points on ∂X , then all orbits converge locally unif. to a single point of ∂X .

3) (Finite-order) f has two distinct fixed points. Then f is periodic.

4) (Rotation) f is conjugate to irrational rotation on $\mathbb{D}$, $\mathbb{D} \setminus \{0\}$, or an annulus $\{1 < |z| < e^{2\pi n}\}$

Proof of $B \Rightarrow A$: $f(U) \subset U$ hyp. f is a comp.

1) Fixed point is attracting by Schwarz,
 U is an attracting basin

3) Can't happen.

4) $U \cong \mathbb{D}$: Siegel disk

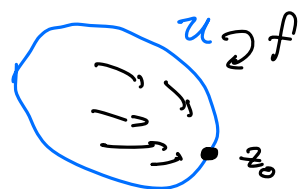
$U \cong \mathbb{D} \setminus \{0\}$: lift to $\mathbb{D}$

$U \cong \{1 < |z| < e^{2\pi n}\}$: Herman ring

2) $\exists z_0 \in \partial U \subset \hat{\mathbb{C}}$ s.t. $\forall z \in U$, $f^n(z) \rightarrow z_0$, $f'(p) = \lambda$,

must have $|\lambda| = 1$

Claim. $\lambda = 1$:

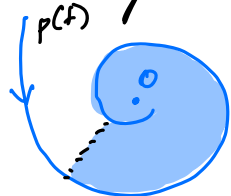


Lemma (Fatou lemma) Let $p: [0, \infty) \rightarrow U$ be

a cont. path s.t. $f(p(t)) = p(t+1)$ and

$p(t) \rightarrow z_0$ as $t \rightarrow \infty$. Then $|\lambda| < 1$ or $\lambda = 1$.

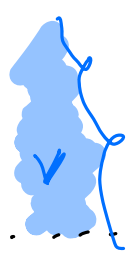
Proof. $|\lambda| \leq 1$. Suppose $|\lambda| = 1$ but $\lambda \neq 1$: $\lambda = e^{it}$



z close: Region would be invariant and hence strictly contracting.

$\uparrow z = e^w$. Channel z_0 : $f(z) = \lambda z + o(1)$

w . Lift $z = e^w$, $f(e^w) = e^{\tilde{F}(w)}$,



$$\tilde{F}(w) = w + it + o(1).$$

$\tilde{F}$ is inj for large w since f' is.

Lift p : $p(t) = e^{\tilde{P}(t)}$, $\operatorname{Re} \tilde{P}(t) \rightarrow -\infty$,

$$\tilde{P}(t+1) = \tilde{P}(t) + it + o(1)$$

For large enough t ,

$$\operatorname{Re} \tilde{P}(t) \leq x_0, \operatorname{Im} \tilde{P}(t+1) \geq \operatorname{Im} \tilde{P}(t) + i\frac{1}{2}$$

The region V to the left of $\tilde{P}$ will map injectively into itself and $\operatorname{Re}(\tilde{F}^{-1}(z)) \rightarrow -\infty$ in this region.

Since $t \neq 0$, $\exp(x)$ will contain a disk whose center $\rightarrow 0$, and hence z_0 is contained.

Proof of Theorem B.

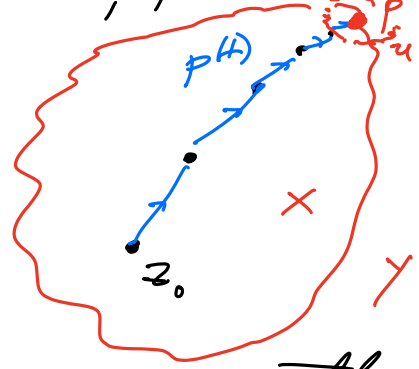
2) Suppose $\exists p$ s.t. $d(f^n(z), z) \rightarrow \infty$



By Pick's theorem, any ball will shrink.

"Oscillator";

• $X \subset Y$, $f: \bar{X} \rightarrow \bar{X}$. $p: [0, \infty) \rightarrow X$, $p(0) = z_0$,
 $p(1) = f(z_0)$, $p(t+1) = f(p(t))$.



• Suppose $\hat{p} \in \partial X$ is an accumulation point of $p(t)$. Sim. hyp.

metric on X shrinks near ∂X compared w/ the metric on Y , for any open $U \subset Y$ st. $U \ni \hat{p}$,
 $U \ni p(t)$ for t large. $\Rightarrow f(\hat{p}) = \hat{p}$.

Lemma The set of limit points of the curve
 $p: [0, \infty) \rightarrow X$ not be connected.

• Final part of $f|_{\partial X}$ are finite, so only $\hat{p}$,
uniquely $\rightarrow \hat{p}$. End of (2)

• Now suppose $d(f^n(z_0), z_0) \rightarrow \infty$:

$$z_i := f^{u_i}(z_0) \rightarrow z_\infty. \quad f^{u_i - u_{i-1}}(z_{i-1}) = z_i$$

• $\{f^{u_i - u_{i-1}}: X \rightarrow X\}$ is normal $\Leftrightarrow f^{u_i - u_{i-1}} \rightarrow g$
 $g(z_\infty) = z_\infty$

Pick $\begin{matrix} \text{or} \\ \text{I) } f \text{ is strictly contracting} \\ \text{II) } f \text{ is a local isometry} \end{matrix}$

I) $\Rightarrow g$ is strictly contracting

• $f \circ g = g \circ f, \Rightarrow f(z_\infty) = g(f(z_\infty))$. But since

g contract, $f(z_\infty) = z_\infty$, and then 1).

II) $\Rightarrow g$ is a local isometry, $|g'(z_\infty)|_{h,p} = 1$.

Take a subsequence of iterates $g^{n_i} \rightarrow h$ so that $h'(z_\infty) = 1$. h is a limit of iterates of f .

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\tilde{h}} & \mathbb{D} \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & X \end{array} \quad \cdot \text{ Lift. By Schwarz, } \tilde{h} = \text{id}, h = \text{id}.$$

$\Rightarrow f$ is inv and surj

$\Rightarrow f$ is an iso $X \xrightarrow{\sim} X$.

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{\tilde{f}} & \mathbb{D} \\ \pi \downarrow & & \downarrow \\ X & \xrightarrow{f} & X = \bigcup_{\Gamma} \mathbb{D} \end{array} \quad \cdot \text{ Lift } f. \tilde{f} \text{ is iso, Nielsen.}$$

Also $f^{u_k} \rightarrow \text{id}$, $\tilde{f}^{u_k} \approx \gamma_k / \text{conjugate}$

for some $\gamma_k \in \Gamma$.

• Since $\tilde{f}^{u_k}$ is a lift of f^{u_k} , $\exists$ homeomorphism

$\phi_k : \Gamma \rightarrow \Gamma$ s.t. $\tilde{f}^{u_k} \circ \gamma = \phi_k(\gamma) \circ \tilde{f}^{u_k}$. } *Aside*

$$\Rightarrow \phi_k(\gamma) = \tilde{f}^{u_k} \circ \gamma \circ \tilde{f}^{-u_k}$$

$$\gamma_k^{-1} \phi_k(\gamma) \gamma_k = \underbrace{\gamma_k^{-1} \tilde{f}^{u_k}}_{\approx \text{id}} \gamma \underbrace{\tilde{f}^{-u_k} \gamma_k}_{\approx \text{id}} \approx \gamma$$

• Since Γ is discrete $\Rightarrow \phi_k(\gamma) = \gamma_k \gamma \gamma_k^{-1}$,

and $\gamma_k^{-1} \tilde{f}^{u_k} : \Gamma$ commute.

• s.t. $\gamma_k^{-1} \tilde{f}^{u_k} = \text{id}$, then f is finite - order.

✓ *Obtain:*

- Max commutator subgroup $C(\gamma)$ of $\gamma \in \Gamma$. $\Rightarrow \gamma_n^{-1} \tilde{f}^n \in C(\gamma)$
 full γ , or Γ is commutator, $C(\gamma) = C(\Gamma) \Rightarrow \tilde{f}^n \in C(\Gamma)$.

i) $\tilde{f}$ is elliptic: $\tilde{f} \in f$ are inv. isotat,

π is an inv.

ii) $\tilde{f}$ hyp: $\tilde{f}$ is inv. isotat translations,
 f is inv. isotat on an orbifold

iii) $\tilde{f}$ parabolic: $\tilde{f}$ is an inv.

parabolic translations, f
 is inv. isotat on $D \times \{0\}$.

